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Inflation, Extra Dimensions, and $\Omega_\Lambda \approx 1$

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An $(N + 1)$-dimensional quantum mechanical model for the origin of the universe results in a 58e-fold inflation and a cosmological constant/vacuum energy density with $\Omega_\Lambda \approx 1$.

Turner [1] claims astrophysical observations require a cosmological constant/vacuum energy density with $\Omega_\Lambda \approx 1$, and notes that few models produce a cosmological constant of this magnitude. This paper considers an $(N + 1)$-dimensional quantum mechanical model for the origin of the universe from a quantum fluctuation [2]. In the model, inflation produces a cosmological constant with $\Omega_\Lambda \approx 1$. The model is developed [2] by combining general relativity with ordinary quantum mechanics [3] and additional compact dimensions, as suggested by superstring theories or M-theory. It is an attempt at a quantum theory of space, but it is obviously not a quantum field theory of gravity.

In the model, the size of the extra dimensions corresponds to a gauge singlet scalar field [4] that is constant throughout our three-dimensional space and drives the inflation of three-dimensional space. The radius of curvature of a closed Friedmann universe containing a scalar field ϕ satisfies [5]

$$\left(\frac{dR}{dt}\right)^2 - \left(\frac{8\pi G}{3}\right)\left[\varepsilon_r\left(\frac{R_0}{R}\right)^4 + \varepsilon_m\left(\frac{R_0}{R}\right)^3 + \varepsilon_\phi\right]\left(\frac{R}{c}\right)^2 = -c^2$$

where ε_r, ε_m, and R_0 are respectively the radiation density, matter density, and radius of curvature of the universe today, and $\varepsilon_\phi = \frac{1}{2}(\partial\phi/\partial t)^2 + V_\phi(\phi)$.

The model is formulated in an N-dimensional curvature space describing the curvature of a homogeneous N-dimensional physical space, where N must

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be greater than six. The curvature space has two subspaces related to the Friedmann universe and the compact dimensions. The coordinate in each dimension of a state in the curvature space is the radius of curvature of the corresponding dimension of that state in the N-dimensional physical space. For the universe to begin from a quantum fluctuation, all total quantum numbers must be zero. When the total energy and total angular momentum in curvature space is zero, the Schrödinger equation for the N-dimensional radius of curvature is $-(\hbar^2/2m)\nabla^2_{\Re}\Psi + V_{\Re}\Psi = 0$, where $\Re$ is the magnitude of an N-dimensional vector $\vec{\Re}$ and m is an effective mass. Let $\Re^2 = R^2 + r^2$, where R is the radial coordinate in the three-dimensional subspace describing the curvature of the isotropic Friedmann universe and r is the radial coordinate in the $(n = N - 3)$-dimensional subspace describing the curvature of the compact dimensions. Separation of $V_{\Re}$ into terms involving only R and terms involving only r is suggested by the fact that the Friedmann equation without explicit dependence on compact dimensions has been a useful model for our $(3 + 1)$-dimensional universe. If $V_{\Re} = V_R + V_r$, separating variables gives

$$\left[\frac{1}{\Psi(R)}\frac{-\hbar^2}{2m}\nabla^2_R\Psi(R) + V_R\right] + \left[\frac{1}{\Psi(r)}\frac{-\hbar^2}{2m}\nabla^2_r\Psi(r) + V_r\right] = 0$$

where each bracket must be constant. These constants are denoted $-E$ and E, respectively. The Schrödinger equation for the radius of curvature R is the quantum mechanical analog of the Friedmann equation [3]. For a radiation-dominated universe with $\varepsilon_\phi = 0$, this Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dR^2}\psi - \frac{4\pi mG}{3c^2}\frac{\varepsilon_r R_0^4}{R^2}\psi = -\frac{\hbar^2}{2m}\frac{d^2}{dR^2}\psi - \frac{mG}{2}\frac{A}{R^2}\psi = -\frac{mc^2}{2}\psi \qquad (1)$$

An S-wave Schrödinger equation must be used for the compact dimensions to make the total N-dimensional "angular momentum" in the curvature space zero. Writing $\Psi = R^{-1}\,\psi(R)r^{-(n-1)/2}\psi'(r)$, we obtain the separated Schrödinger equation

$$\left[\frac{1}{\psi(R)}\frac{-\hbar^2}{2m}\frac{d^2\psi(R)}{dR^2} + V_R\right] + \left[\frac{1}{\psi'(r)}\frac{-\hbar^2}{2m}\frac{d^2\psi'(r)}{dr^2} + \left(\frac{\hbar^2(n-1)(n-3)}{8mr^2} + V_r\right)\right] = 0$$

In this model, the potential V_r constraining the size of the compact dimensions is parametrized as $V_r = kr^\alpha$. Choosing $\alpha = 4$ produces a universe similar to our own [2]. If $k = \hbar^2(n-1)(n-3)/(16m\delta^6)$, the minimum of the effective potential $\hbar^2(n-1)(n-3)/(8mr^2) + kr^4$ for the compact dimensions is at the Planck length δ.

Initially, gravity and the strong-electroweak (SEW) force had equal strength, so $G_i m_p^2/\hbar c = 1$, the gravitational constant was $G_i = 1.70 \times 10^{38} G$, the Planck length was $\delta_i = 2.11 \times 10^{-14}$ cm, and the Planck mass M_i equaled the proton mass m_p. The minimum of the effective potential for the compact dimensions was initially at $r = \delta_i$. The effective potential for the compact dimensions can be approximated by a harmonic oscillator potential near its minimum, so the ground-state energy of the effective potential is

$$\frac{\hbar^2}{2m\delta^2}\left[\frac{3}{8}(n-1)(n-3) + \sqrt{\frac{3}{2}(n-1)(n-3)}\right] = \frac{\hbar^2\beta^2}{2m\delta^2}$$

In the initial state, $mc^2/2 = \hbar^2\beta^2/2m\delta_i^2$ and $m = \beta\sqrt{\hbar c/G_i}$, where $\sqrt{\hbar c/G_i} = m_p$ was the initial Planck mass. If, for example, the fundamental theory of the forces governing the universe requires 9, 10, or 11 spatial dimensions (as might be true in the ardently sought superstring theory or M-theory), $3.22 < \beta < 4.51$.

The universe began with a quantum fluctuation into the ground state of a radiation-dominated universe obeying the Schrödinger equation (1), with gravitational constant G_i, radius $\langle R\rangle = \delta_i$ and energy $-\frac{1}{2}\beta m_p c^2$, and the ground state of the compact dimensions with radius $\langle r\rangle = \delta_i$ and energy $\frac{1}{2}\beta m_p c^2$. This initial state had $dR/dt = dr/dt = 0$. Both the Friedmann universe and the compact dimensions were in the ground state and could not lose energy to the other. So the two subsystems were effectively decoupled and ε_ϕ was zero.

After the initial state arose from a quantum fluctuation, a quantum tunneling transition occurred from the initial state to another state with zero total curvature energy, where the minimum of the effective potential in the compact dimensions is at $r = \delta$, $k = \hbar^2(n-1)(n-3)/(16\beta M\delta^6)$, and the ground-state energy of the compact dimensions is $\frac{1}{2}\beta Mc^2$. This post-transition state was the beginning of today's universe, where the size of the compact dimensions corresponds to a scalar field ϕ that is constant throughout the Friedmann universe. The scalar field $\phi = 0$ when $r = \delta_i$, if it is related to the size of the compact dimensions [4] by $\phi = (\xi/\delta_i)\sqrt{\hbar/c}\,\ln(\delta_i/r)$. Here ξ is a real number to be obtained from the fundamental theory of forces.

Immediately after the transition, $\langle R\rangle = \langle r\rangle = \delta_i$ and $dR/dt = dr/dt = 0$. The compact dimensions were in a highly excited state of the effective potential V_r, with wave packet localized at the classical turning radius $r = \delta_i$, and curvature energy

$$E' = \frac{\hbar^2(n-1)\,(n-3)}{16\beta M\delta^2}\left(\frac{\delta_i}{\delta}\right)^4 = \frac{(n-1)(n-3)}{\beta}\,3.45 \times 10^{91}\,\frac{\text{g cm}^2}{\text{sec}^2}$$

At transition, the Friedmann universe satisfied the Schrödinger equation for a universe containing radiation and a scalar field:

$$-\frac{\hbar^2}{2m_\phi}\frac{d^2}{dR^2}\psi - \frac{4\pi m_\phi G_\phi}{3}(\varepsilon_r + \varepsilon_\phi)\left(\frac{R}{c}\right)^2\psi = -E'\psi$$

or

$$-\frac{\hbar^2}{2m_\phi}\frac{d^2}{dR^2}\psi - \frac{4\pi\delta_\phi}{3}\left(\frac{A'}{R^2} + \varepsilon_\phi R^2\right)\psi = -E'\psi \qquad (2)$$

Here, ε_ϕ models the effect of the compact dimensions on the Friedmann universe and A' is proportional to the number of photons in the universe [6]. The top of the effective potential in equation (2) is at $R^4_{\text{peak}} = A'/\varepsilon_\phi$. The transition energy E' coincided with the top of the potential, so the transition resulted in a state with wave packet centered at $R = \delta_i$ in unstable equilibrium with $dR/dt = 0$. At transition, $\varepsilon_\phi = A'/\delta_i^4$, so $\varepsilon_r = \varepsilon_\phi$ and

$$\frac{8\pi m_\phi G_\phi A'}{3c^2\delta_i^2} = \frac{\hbar^2(n-1)(n-3)}{16\beta M\delta^2}\left(\frac{\delta_i}{\delta}\right)^4$$

Because the bottom of the effective potential in the compact dimensions is now at $r = \delta$, $V_\phi(0) \neq 0$ at transition. When the curvature energy of the compact dimensions dropped to the ground-state energy $\frac{1}{2}\beta Mc^2$, the curvature energy of the Friedmann universe was raised to the Einstein value $-\frac{1}{2}\beta Mc^2$. The scalar field ϕ increased from zero to ϕ_f as the characteristic size of the compact dimensions decreased from δ_i to δ, G decreased from G_i to its present value, and the Planck mass increased from m_p to its present value. When $R > R_{\text{peak}}$, the ε_ϕ term in equation (2) dominated and the Friedmann universe inflated. The radiation energy density ε_r increased as the energy in the scalar field converted to radiation. At the end of inflation, $\varepsilon_\phi \ll \varepsilon_{\text{rad}}$, and the radiation-dominated universe satisfied equation (1) with $A = (\hbar/3c)\,(\delta_i/\delta)^6$. Then, $V_R = 3.8(m/2)(10^{69}/R^2)$ g cm^2/sec^2, near the approximate value $V_R = 5.6$ (m/2)$(10^{71}/R^2)$ g cm^2/sec^2 estimated for our universe [2].

The model can be used to estimate the vacuum energy density of today's universe (and the extent of inflation) by assuming that the curvature energy of the compact dimensions dropped instantaneously to the ground state when the wave packet reached $\langle r\rangle = \delta_i$, and noting that a spatially constant scalar field has only one degree of freedom. In what follows, the ratio E'/E_g of the transition energy to the ground state energy is taken as 5×10^{75}. (The ratio 5.76×10^{75} for 10 space dimensions).

At the end of inflation, the temperature of the universe was T_e, and the compact dimensions were in the ground state with curvature energy E_g. At that time, $\dot{\phi}^2 = 0$ and the energy density of the spatially constant scalar field in the Friedmann universe was $\varepsilon_\phi = V_\phi(\phi_f)$ thereafter. Immediately after the quantum transition from the symmetric initial state (where both the compact dimensions and the Friedmann universe were in their ground states) the curvature energy and the entropy of the compact dimensions was 5×10^{75} larger than at the end of inflation. The energy density of the spatially constant scalar field in the Friedmann universe was $\varepsilon_\phi = V_\phi(0)$. When the curvature energy of the compact dimensions dropped to the ground state, the entropy of the compact dimensions was lowered by a factor of 0.2×10^{-75} and the entropy of the Friedmann universe was raised by a factor of 5×10^{75}. Then, the temperature of the Friedmann universe was $(5 \times 10^{75})^{1/3}\, T_e = 1.7 \times 10^{25}\, T_e$. As the wave packet described by equation (2) moved away from the peak of the effective potential at $R = \delta_t$, the Friedmann universe expanded exponentially and isentropically (driven by the scalar field energy) until inflation ended when $\dot{\phi}^2 = 0$ and the temperature of the universe reached T_e. Thus, the scale factor of the Friedmann universe increased by a factor of 1.7×10^{25} during inflation. This 58*e*-fold inflation is within the limits set by fluctuations in the microwave background radiation [7].

From the Schrödinger equation (2) at transition, the potential term was $a^3 \nu E'/a^3 \nu = E'$ when the Friedmann universe was at the unstable equilibrium point immediately after the quantum transition from the initial state. Here $a = \delta$, is the scale factor, ν is the unit coordinate volume, and E'/ν is the energy density (i.e., the energy in a unit coordinate volume [8]). So the energy density in the scalar field in the Friedmann universe at the start of inflation was $V_\phi(0) = E'/2 \approx 10^{92}$ g cm^{-1} sec^{-2}. A spatially constant scalar field has only one degree of freedom, and the energy density in the scalar field is proportional to the fourth power of the temperature [9]. So the energy density of the scalar field at the end of inflation was

$$\varepsilon_e \approx \frac{T_e^4}{(1.7 \times 10^{25} T_e)^4}\, 10^{92} \text{ g cm}^{-1} \text{ sec}^{-2} \approx 10^{-9} \text{ g cm}^{-1} \text{ sec}^{-2}$$

The above argument can be restated as follows. Suppose the universe were to be "run in reverse" from the moment when inflation ended and the temperature of the universe was T_e. Then isentropic compression to a temperature $(5 \times 10^{75})^{1/3}\, T_e = 1.7 \times 10^{25}\, T_e$ would be necessary to raise the entropy density of the Friedmann universe to the value it had immediately after the curvature energy of the compact dimensions fell to the ground state, and raise the energy density of the scalar field to the value it had at the time of that transition.

At the end of inflation, after the scalar field stops decaying into radiation and $\dot{\phi}^2 = 0$, the scalar field equation of state is $p = -\varepsilon$. So the energy density of the scalar field in the Friedmann universe did not change after inflation [6]. If $H_0 = 65$ km sec^{-1} Mpc^{-1} [1], then the critical density is $\rho_c = 3H_0^2/8\pi G = 7.9 \times 10^{-30}$ g cm^{-3} and the critical energy density is $\varepsilon_c = \rho_c c^2 = 7.11 \times 10^{-9}$ gm cm^2 sec^{-2} cm^{-3}. Thus, in this model, $\Omega_\Lambda = \varepsilon_e/\varepsilon_c \approx 1$.

The decay rate of the scalar field to radiation, to be obtained from the fundamental theory of forces, is constrained by the requirement that density fluctuations in the cosmic microwave background have the correct magnitude [10]. The matter density produced in the radiation-dominated universe after inflation, and thus Ω_m, must also be calculated from the GUT obtained from the fundamental theory of forces.

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